Geometric and probabilistic results for the observability of the wave equation

Emmanuel Humbert, Yannick Privat, Emmanuel Trélat

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Abstract

Given any measurable subset ω of a closed Riemannian manifold (M, g) and given any $T > 0$, we define $\ell^T(\omega) \in [0, 1]$ as the smallest average time over $[0, T]$ spent by all geodesic rays in ω. This quantity appears naturally when studying observability properties for the wave equation on M, with ω as an observation subset: the condition $\ell^T(\omega) > 0$ is the well known *Geometric Control Condition*.

In this article we establish two properties of the functional ℓ^T, one is geometric and the other is probabilistic.

The first geometric property is on the maximal discrepancy of ℓ^T when taking the closure. We may have $\ell^T(\mathring{\omega}) < \ell^T(\overline{\omega})$ whenever there exist rays grazing ω and the discrepancy between both quantities may be equal to 1 for some subsets ω. We prove that, if the metric g is C^2 and if ω satisfies a slight regularity assumption, then $\ell^T(\overline{\omega}) \leqslant \frac{1}{2}\left(\ell^T(\mathring{\omega}) + 1\right)$. We also show that our assumptions are essentially sharp; in particular, surprisingly the result is wrong if the metric g is not C^2. As a consequence, if ω is regular enough and if $\ell^T(\overline{\omega}) > 1/2$ then the Geometric Control Condition is satisfied and thus the wave equation is observable on ω in time T.

The second property is of probabilistic nature. We take $M = \mathbb{T}^2$, the flat two-dimensional torus, and we consider a regular grid on it, a regular checkerboard made of n^2 square white cells. We construct random subsets ω_ε^n by darkening each cell in this grid with a probability ε. We prove that the random law $\ell^T(\omega_\varepsilon^n)$ converges in probability to ε as $n \to +\infty$. As a consequence, if n is large enough then the Geometric Control Condition is satisfied almost surely and thus the wave equation is observable on ω_ε^n in time T.

1 Introduction and main results

Let (M, g) be a closed connected Riemannian manifold. We denote by Γ the set of geodesic rays, that is, the set of projections onto M of Riemannian geodesic curves in the co-sphere bundle S^*M. Given any $T > 0$ and any Lebesgue measurable subset ω of M, we define

$$\ell^T(\omega) = \inf_{\gamma \in \Gamma} \frac{1}{T} \int_0^T \chi_\omega(\gamma(t))\, dt. \tag{1}$$

Here, χ_ω is the characteristic function of ω, defined by $\chi_\omega(x) = 1$ if $x \in \omega$ and $\chi_\omega(x) = 0$ if $x \in M \setminus \omega$. The real number $\ell^T(\omega)$ is the smallest average time over $[0, T]$ spent by all geodesic

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rays in ω. The condition

$$\ell^T(\omega) > 0$$

means that all geodesic rays, propagating in M, meet ω within time T. This condition, usually called *Geometric Control Condition* (in short, GCC), is related to observability properties for the wave equation

$$\partial_{tt} y - \triangle_g y = 0 \qquad \text{in } (0,T)\times M \tag{2}$$

where $\triangle_g$ is the Laplace-Beltrami operator on M for the metric g.

More precisely, denoting by dx_g the canonical Riemannian volume, we define the observability constant $C_T(\omega) \geqslant 0$ as the largest possible nonnegative constant C such that the inequality

$$\int_0^T \int_\omega |y(t,x)|^2 \, dx_g \, dt \geqslant C \left(\|y(0,\cdot)\|^2_{L^2(M)} + \|\partial_t y(0,x)\|^2_{H^{-1}(M)} \right) \tag{3}$$

is satisfied for any solution y of (2), that is,

$$C_T(\omega) = \inf \left\{ \int_0^T \int_\omega |y(t,x)|^2 \, dx_g \, dt \;\mid\; \|(y(0,\cdot), \partial_t y(0,\cdot))\|_{L^2(M)\times H^{-1}(M)} = 1 \right\}.$$

When $C_T(\omega) > 0$, the wave equation (2) is said to be *observable* on ω in time T, and when $C_T(\omega) = 0$ we say that observability does not hold for (ω, T).

It has been proved in [1, 9] that, for ω open, observability holds if the pair (ω, T) satisfies GCC, i.e., if $\ell^T(\omega) > 0$. In other words, if ω is open and satisfies $\ell^T(\omega) > 0$ then the wave equation (2) is observable on ω in time T.

The converse is not true: GCC is not a necessary condition for observability. It is shown in [8] that, if $M = \mathbb{S}^2$ (the unit sphere in $\mathbb{R}^3$ endowed with the restriction of the Euclidean structure), if ω is the open Northern hemisphere, then $\ell^T(\omega) = 0$ for every $T > 0$, and however one has $C_T(\omega) > 0$ for every $T > \pi$. The latter fact is established by an explicit computation exploiting symmetries of solutions. This failure of the functional ℓ^T to capture the observability property is due to the existence, here, of a very particular geodesic ray which is *grazing* the open set ω, namely, the equator. In this example, consideering the closure $\overline{\omega}$ of ω, it is interesting to observe that $\ell^T(\overline{\omega}) = 0$ for every $T \leqslant \pi$ (take a geodesic ray contained in the closed Southern hemisphere) and $\ell^T(\overline{\omega}) > 0$ for every $T > \pi$, with $\ell^T(\overline{\omega}) = \frac{1}{2}$ when $T \geqslant 2\pi$. The latter equality is in contrast with $\ell^T(\omega) = 0$: there is thus a discrepancy $1/2$ in the value of ℓ^T for $T \geqslant 2\pi$ when taking the closure of ω. In this specific case, this discrepancy is caused by the equator, which is a geodesic ray grazing the open subset ω.

Our first main result below shows that $1/2$ is actually the maximal possible discrepancy.

1.1 A geometric result on the maximal discrepancy of ℓ^T

In general, one can always find subsets ω for which the difference $\ell^T(\overline{\omega}) - \ell^T(\mathring{\omega})$ is arbitrary close to 1. Surprisingly, under slight regularity assymptions, this maximal discrepancy is $1/2$ only.

Theorem 1. *Let $T > 0$ be arbitrary and let ω be a measurable subset of M. We make the following assumptions:*

(i) The metric g is at least of class C^2.

(ii) ω is an embedded C^1 submanifold of M with boundary if $n \geqslant 3$ and is piecewise C^1 if $n = 2$.

Then

$$\ell^T(\overline{\omega}) \leqslant \frac{1}{2}\left(\ell^T(\mathring{\omega}) + 1\right). \tag{4}$$

We give more details and a number of comments on this theorem in Section 2.

At the opposite, as an obvious remark, if there is no geodesic ray grazing ω (more precisely, if there is no geodesic ray having a contact of infinite order with $\partial\omega = \overline{\omega} \setminus \mathring{\omega}$) then $\ell^T(\overline{\omega}) = \ell^T(\mathring{\omega})$.

In more general, the existence of such *grazing rays*, which are rays having a contact of infinite order with $\partial\omega = \overline{\omega} \setminus \mathring{\omega}$ and may involve an arc entirely contained in $\partial\omega$, adds a serious difficulty to the analysis of observability (see [1]).

It is noticeable that, if one replaces the characteristic function χ_ω of ω by a continuous function a, in the integral at the left-hand side of (3) (i.e., $\int_0^T \int_M a(x)|y(t,x)|^2 \, dx_g \, dt$) as well as in the definition (1) of the functional ℓ^T, this difficulty disappears and the condition $\ell^T(a) > 0$ becomes a necessary and sufficient condition for observability of (2) on ω in time T (see [4]).

By the way, for completeness, we provide in Appendix A some semi-continuity properties of the functional ℓ^T, which may be of interest for other purposes.

The issue of the observability on a general measurable subset $\omega \subset M$ has remained widely open for a long time. Recent advances have been made, which we can summarize as follows. It has been established in [7] that observability on a measurable subset ω in time T is satisfied if and only if $\alpha_T(\omega) > 0$. The quantity $\alpha_T(\omega)$, defined in [7] as the limit of highfrequency observability constants, is however not easy to compute and we have, in general, the inequality $\ell^T(\mathring{\omega}) \leqslant \alpha_T(\omega) \leqslant \ell^T(\overline{\omega})$. In particular, the condition $\ell^T(\omega) > 0$ becomes a necessary and sufficient condition for observability as soon as there are no geodesic rays grazing ω. It has also been shown in [7] that $\lim_{T\to+\infty} \frac{C_T(\omega)}{T}$ is the minimum of two quantities, one of them being $\ell^T(\overline{\omega})$ and the other being of a spectral nature.

We have the following corollary of Theorem 1, using the fact that, since $\mathring{\omega}$ is open, the condition $\ell^T(\mathring{\omega}) > 0$ implies observability for $(\mathring{\omega}, T)$, and thus $C_T(\omega) \geqslant C_T(\mathring{\omega}) > 0$.

Corollary 1. *Under the assumptions of Theorem 1, if $\ell^T(\overline{\omega}) > 1/2$ then $\ell^T(\mathring{\omega}) > 0$ and thus the wave equation (2) is observable on ω in time T.*

Note that Corollary 1 does not apply to the (limit) case where $M = \mathbb{S}^2$ and ω is the open Northern hemisphere. It does neither apply to the case where M is the two-dimensional torus and ω is a half-covering open checkerboard on it, as in [3, 5] (see next section). Indeed, in these two cases, we have $\ell^T(\omega) = 0$ for every $T > 0$ but $C_T(\omega) > 0$ (i.e., we have observability) for T large enough. This is due to the fact that trapped rays are the weak limit of Gaussian beams that oscillate *on both sides* of the limit ray, spreading on one side and on the other a sufficient amount of energy so that indeed observability holds true. In full generality, having information on the way that semi-classical measures, supported on a grazing ray, can be approached by highfrequency wave packets such as Gaussian beams, is a difficult question. In the case of the sphere, symmetry arguments give the answer (see [8]). In the case of the torus, a much more involved analysis is required, based on second microlocalization arguments (see [3, 5]).

Anyway, Corollary 1 can as well be applied for instance to any kind of checkerboard domain ω on the two-dimensional torus, as soon as the measure of ω is large enough so that $\ell^T(\overline{\omega}) > 1/2$.

Since the case of checkerboards (in dimension two) is interesting and challenging, following a question by Nicolas Burq, in the next section we investigate the case of random checkerboards on the flat torus and we establish our second main result.

1.2 A probabilistic result for random checkerboards on the flat torus

In this section, we take $M = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ (flat torus) which is identified to the square $[0,1]^2$, class of equivalence of $\mathbb{R}^2$ under the identifications $(x,y) \sim (x+1,y) \sim (x,y+1)$, inheriting of the Euclidean metric. Given any subset A of M, we denote by $|A|$ the (two-dimensional) Lebesgue measure of A.

We consider a regular grid $\mathcal{G}^n = (c_{ij})_{1 \leqslant i,j \leqslant n}$ in the square, like a checkerboard, made of $n \times n$ closed squares:

$$[0,1]^2 = \bigcup_{i,j=1}^{n} c_{ij} \qquad \text{with} \qquad c_{ij} = [i-1, i] \times [j-1, j] \qquad \forall (i,j) \in \{1, \ldots, n\}^2.$$

Defining $c_{i'j'}$ in the same way for all $(i', j') \in \mathbb{Z}^2$, we identify the square $c_{i'j'}$ to the square c_{ij} of the above grid with $(i,j) \in \{1, \ldots, n\}^2$ such that $i = i' \mod n$ and $j = j' \mod n$.

Construction of random checkerboards. Let $\varepsilon \in [0,1]$ be arbitrary. Considering that all squares in the grid are initially white, we construct a random checkerboard by randomly darkening some squares in the checkerboard as follows: for every $(i,j) \in \{1, \ldots, n\}^2$, we darken the square c_{ij} of the grid with a probability ε. All choices are assumed to be mutually independent. In other words, we make a selection of squares (that are paint in black) in the grid by considering n^2 independent Bernoulli random variables denoted $(X_{ij})_{1 \leqslant i,j \leqslant n}$, each of them with parameter ε. The total number of black squares follows therefore the binomial law $B(n^2, \varepsilon)$.

We denote by ω_ε^n the resulting (closed) subset of $[0,1]^2$ that is the union of all black squares (see Figure 1). Given any fixed $T > 0$ and $\varepsilon \in (0,1]$, our objective is to understand how well the

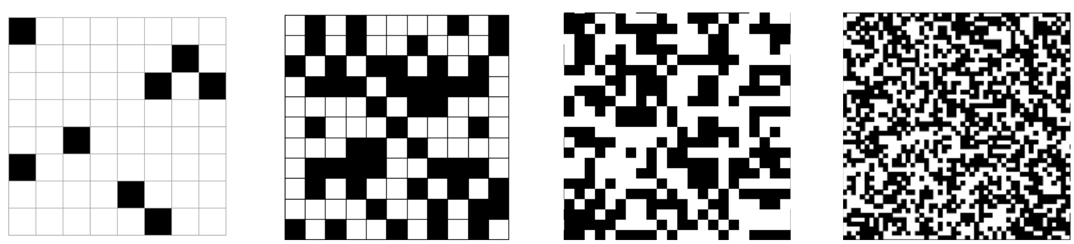

Figure 1: Some examples of random checkerboards. The random subset $\omega_\varepsilon^n \subset [0,1]^2$ is the union of black squares.

random set ω_ε^n is able to capture all geodesic rays propagating in $M \simeq [0,1]^2$, in finite time T. In other words, we want to study the random variable $\ell^T(\omega_\varepsilon^n)$. Of course, the random variable $|\omega_\varepsilon^n|$ follows the law $\frac{1}{n^2} B(n^2, \varepsilon)$ and thus its expectation is equal to ε, and so, when ε is small, ω_ε^n covers only a small area in $[0,1]^2$. And yet, our second main result below shows that, for n large, almost all such random sets meet all geodesic rays within time T.

Theorem 2. *Given any $T > 0$ and any $\varepsilon \in [0,1]$, the random variable $\ell^T(\omega_\varepsilon^n)$ converges in probability to ε as $n \to +\infty$, i.e.,*

$$\lim_{n \to +\infty} \mathbb{P}\left(|\ell^T(\omega_\varepsilon^n) - \varepsilon| \geqslant \delta \right) = 0 \qquad \forall \delta > 0.$$

Theorem 2 is proved in Section 3. As mentioned above, this issue has emerged following a question by Nicolas Burq. In [3, 5], the authors also consider checkerboard domains, as above, but not in a random framework. As a consequence of their analysis, given any $T > 0$, any $\varepsilon \in [0,1]$ and any $n \in \mathbb{N}^*$ fixed, if all geodesic rays of length T, either meet the interior of ω_ε^n (i.e., the interior of some black square), or follow for some positive time one of the sides of a black square on the left and for some positive time one of the sides of a black square (possibly the same) on the right, then $C_T(\omega_\varepsilon^n) > 0$, i.e., the wave equation on the torus $M = \mathbb{T}^2$ is observable on ω_ε^n in time T.

Let $T > 0$ and let $\varepsilon \in (0, 1]$ be arbitrary. According to Theorem 2, for n large enough, almost every subset ω_ε^n (constructed randomly as above) is such that $\ell^T(\omega_\varepsilon^n) > 0$. This implies that every geodesic ray, that is neither horizontal nor vertical, meets the interior of ω_ε^n within time T, and that every horizontal or vertical geodesic ray meets the closed subset ω_ε^n within time T. In the latter case, moreover, by construction of the random set ω_ε^n, the probability that such grazing rays follow for some positive time one of the sides of a black square on the left and for some positive time one of the sides of a black square, converges to 1 as $n \to +\infty$.

All in all, combining Theorem 2, the result of [3, 5] and the above reasoning, we have the following consequence in terms of observability of the wave equation.

Corollary 2. *Given any $T > 0$ and $\varepsilon \in (0, 1]$, may they be arbitrarily small, the probability that the wave equation on the torus $M = \mathbb{T}^2$ is observable on ω_ε^n in time T tends to 1 as $n \to +\infty$.*

In other words, observability in (any) finite time is almost surely true for large n, despite the fact that the measure of ω_ε^n may be very small!

Note that, for $\varepsilon > 1/2$, almost sure observability follows from Corollary 1 (indeed, the random sets constructed above are piecewise C^1 and thus Theorem 1 can be applied). But the result is more striking when $|\omega_\varepsilon^n|$ is small.

Note also Theorem 2 provides an answer to an issue raised in [6], which we formulate in terms of an optimal shape design problem in the next corollary.

Corollary 3. *For every $\varepsilon \in [0, 1]$, we have*

$$\sup_{|\omega| \leqslant \varepsilon} \ell^T(\omega) = \varepsilon$$

where the supremum is taken over all possible measurable subsets ω of $M = \mathbb{T}^2$ having a Lipschitz boundary.

Corollary 3 is proved in Section 3.4.

We finish this section by a comment on possible generalizations of Theorem 2. Some of the steps of its proof remain valid for any closed Riemannian manifold, like the fact that it suffices to prove the theorem for T small and thus, we expect that, to some extent, the result is purely local. However, in some other steps we instrumentally use the fact that we are dealing with a regular checkerboard in the square. Extending the result to general manifolds, even in dimension two, is an open issue.

2 Additional comments and proof of Theorem 1

2.1 Comments on Theorem 1

Theorem 1 states that, given any $T > 0$ and any measurable subset ω of M, we have

$$\ell^T(\overline{\omega}) \leqslant \frac{1}{2}\left(\ell^T(\mathring{\omega}) + 1\right).$$

under the two following sufficient assumptions:) the metric g is C^2; (ii) ω is an embedded C^1 submanifold of M with boundary if $n \geqslant 3$ and is piecewise C^1 if $n = 2$.

Remark 1. Assumption (ii) may be weakened as follows:

- If M is of dimension 2, it suffices to assume that ω is piecewise C^1. More precisely, we assume that ω is a C^1 *stratified* submanifold of M (in the sense of Whitney).

- In any dimension, the following much more general assumption is sufficient: given any grazing ray γ, for almost every $t \in [0,T]$ such that $\gamma(t) \in \partial\omega$, the subdifferential at $\gamma(t)$ of $\partial\omega \cap \gamma(\cdot)^\perp$ is a singleton. This is the case under the (much stronger) assumption that ω be geodesically convex.

Comments. It is interesting to note that the assumptions made in Theorem 1 are essentially sharp. Remarks are in order.

- The inequality (4) gives a quantitative measure of the discrepancy that can happen for ℓ^T when we take the closure of a measurable subset ω or, conversely, when we pass to the interior (this is the sense of Corollary 1). The inequality is sharp, as shown by the example already discussed above: take $M = \mathbb{S}^2$ and ω the open Northern hemisphere; then $\ell^T(\omega) = 0$ for every $T > 0$ and $\ell^{2\pi}(\overline{\omega}) = 1/2$ for $T = 2\pi$. Hence, here, (4) is an equality.

- As a variant, take ω which is the union of the open Northern hemisphere and of a Southern spherical cap, i.e., a portion of the open Southern hemisphere limited by a given latitude $-\varepsilon < 0$. Then we have as well $\ell^T(\omega) = 0$ for every $T > 0$ and $\ell^{2\pi}(\overline{\omega}) = 1/2$ for $T = 2\pi$.

- Note that, taking $\varepsilon = 0$ in the previous example (i.e., ω is the unit sphere $M = \mathbb{S}^2$ minus the equator), we have $\ell^T(\omega) = 0$ and $\ell^T(\overline{\omega}) = 1$ for every $T > 0$ and thus (4) fails. But here, ω is not an embedded C^1 submanifold of M with boundary: Assumption (ii) (which implies local separation between $\mathring{\omega}$ and $M \setminus \overline{\omega}$) is not satisfied. More generally, the result does not apply to any subset ω that is M minus a countable number of rays. This is as well the case when one considers any subset ω that is dense and of empty interior (one has $\ell^T(\mathring{\omega}) = 0$ and $\ell^T(\overline{\omega}) = 1$ for every $T > 0$). This shows that the discrepancy $1/2$ is only valid under some regularity assumptions on ω.

- There is no discrepancy in the absence of geodesic rays grazing ω: if there is no geodesic ray having a contact of infinite order with $\partial\omega = \overline{\omega} \setminus \mathring{\omega}$ then $\ell^T(\overline{\omega}) = \ell^T(\mathring{\omega})$.

- The result fails in general if $\partial\omega$ is piecewise C^1 only, on a manifold M is of dimension $n \geqslant 3$. Here is a counterexample.

  Let γ be a geodesic ray. If $T > 0$ is small enough, it has no conjugate point. In a local chart of coordinates, we have $\gamma(t) = (t, 0, \ldots, 0)$ (see the proof of Theorem 1). Now, using this local chart we define a subset ω of M as follows: the section of $\partial\omega$ with the vertical hyperplane $\gamma(\cdot)^\perp$ is locally equal to this entire hyperplane minus a cone of vertex $\gamma(t)$ with small angle $2\pi\varepsilon > 0$, less than $\pi/4$ for instance (see Figure 2). Now, we assume that, as $t > 0$ increases, these sections rotate with such a speed that, along $[0,T]$, the entire vertical hyperplane is scanned by the section with ω. If the speed of rotation is exactly $T/2\pi$ then it can be proved that $\ell^T(\mathring{\omega}) = 0$ and $\ell^T(\overline{\omega}) = 1 - 2\varepsilon$.

  This example shows that Assumption (ii), or its generalization given in Remark 1, cannot be weakened too much. The idea here is to consider a subset ω such that the section of $\partial\omega$ with the vertical hyperplane $\gamma(\cdot)^\perp$ has locally the shape of the hypograph of an absolute value, which is rotating along $\gamma(\cdot)$.

  Similar examples can as well be designed with checkerboard-shaped domains ω, thus underlining that in [3, 5] it was important to consider checkerboards in dimension 2.

- Surprisingly, the result is wrong if the metric g is not C^2. A counterexample is the following.

  Let M be a pill-shaped two-dimensional manifold given by the union of a cylinder of finite length, at the extremities of which we glue two hemispheres (domain also obtained by rotating

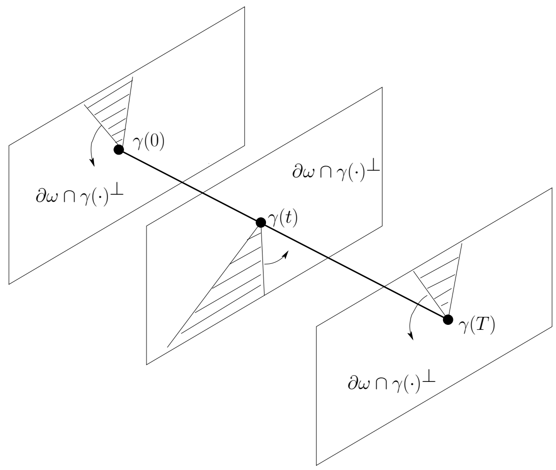

Figure 2: Locally around $\gamma(t)$, $\partial\omega \cap \gamma(t)^\perp$ is the complement of the hatched area.

a 2D stadium in $\mathbb{R}^3$ around its longest symmetry axis; or, take the unit sphere in $\mathbb{R}^3$, cut it at the equator, separate the two hemispheres and glue them with, inbetween, a cylinder of arbitrary length), and endow it with the induced Euclidean metric (see Figure 3). Then the

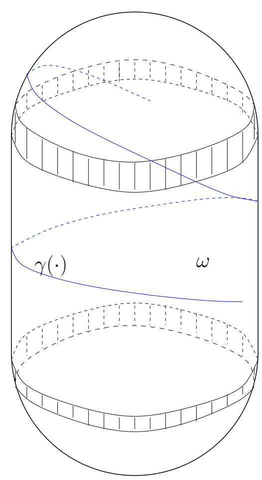

Figure 3: M is pill-shaped and ω is the complement of the hatched area.

metric is not C^2 at the gluing circles. Now, take ω defined as the union of the open cylinder with two open spherical caps (i.e., the union of the two hemispheres of which we remove latitudes between 0 and some $\varepsilon > 0$). Then $\ell^T(\omega) = 0$ for every $T > 0$, because ω does not contain the rays consisting of the circles at the extremities of the cylinder. In contrast, $\ell^T(\overline{\omega})$ may be arbitrarily close to 1 as T is large enough and ε is small enough, and thus (4) fails. This is because any ray of M spending a time π in $M \setminus \omega$ spends then much time over the cylinder.

This shows that Assumption (i) is sharp. In the above example, the metric is only $C^{1,1}$.

The example above is rather counterintuitive. The assumption of a C^2 metric implies in some sense a global result on geodesic rays.

Our proof, given in Section 2.2 hereafter, uses only elementary arguments of Riemannian geometry. It essentially relies on Lemma 2, in which we establish that, given a grazing ray (i.e., a ray propagating in $\partial\omega$), thanks to our assumption on ω we can always construct neighbor rays, one of which being inside ω and the other being outside of ω for all times.

2.2 Proof of Theorem 1

Without loss of generality, we take $\omega \subset M$ open. We will use several well known facts of Riemannian geometry, for which we refer to [2].

Lemma 1. *There exists $\gamma \in \Gamma$ such that $\ell^T(\omega) = \frac{1}{T}\int_0^T \chi_\omega(\gamma(t))\,dt$, i.e., the infimum in the definition (1) of $\ell^T(\omega)$ is reached.*

Proof. Let $(\gamma_k)_{k\in\mathbb{N}}$ be a sequence of geodesic rays such that $\frac{1}{T}\int_0^T \chi_\omega(\gamma_k(t))\,dt \to \ell^T(\omega)$. By compactness of geodesics, $\gamma_k(\cdot)$ converges uniformly to some geodesic ray $\gamma(\cdot)$ on $[0,T]$.

Let $t \in [0,T]$ be arbitrary. If $\gamma(t) \in \omega$ then for k large enough we have $\gamma_k(t) \in \omega$, and thus $1 = \chi_\omega(\gamma(t)) \leqslant \chi_\omega(\gamma_k(t)) = 1$. If $\gamma(t) \in M \setminus \omega$ then $0 = \chi_\omega(\gamma(t)) \leqslant \chi_\omega(\gamma_k(t))$ for any k. In all cases, we have obtained the inequality

$$\chi_\omega(\gamma(t)) \leqslant \liminf_{k\to+\infty} \chi_\omega(\gamma_k(t))$$

for every $t \in [0,T]$.

By the Fatou lemma, we infer that

$$\ell^T(\omega) \leqslant \frac{1}{T}\int_0^T \chi_\omega(\gamma(t))\,dt \leqslant \frac{1}{T}\int_0^T \liminf_{k\to+\infty} \chi_\omega(\gamma_k(t))\,dt \leqslant \liminf_{k\to+\infty} \frac{1}{T}\int_0^T \chi_\omega(\gamma_k(t))\,dt = \ell^T(\omega).$$

The lemma follows. □

If the ray γ given by Lemma 1 is not grazing, i.e., if $\int_0^T \chi_{\partial\omega}(\gamma(t))\,dt = 0$, then $\int_0^T \chi_\omega(\gamma(t))\,dt = \int_0^T \chi_{\overline{\omega}}(\gamma(t))\,dt$ and thus $\ell^T(\overline{\omega}) \leqslant \frac{1}{T}\int_0^T \chi_{\overline{\omega}}(\gamma(t))\,dt \leqslant \ell^T(\omega)$ and hence $\ell^T(\overline{\omega}) = \ell^T(\omega)$. So in this case there is nothing to prove.

In what follows we assume that the ray γ given by Lemma 1 is grazing, i.e., $\int_0^T \chi_{\partial\omega}(\gamma(t))\,dt > 0$. Assume that $\gamma(t) = \pi \circ \varphi_t(x_0, \xi_0)$ with $x_0 \in M$ and $\xi_0 \in S^*_{x_0}M$. Here, $S^*_{x_0}M$ denotes the unit cotangent bundle at x_0 (i.e., $\|\xi_0\|_{g^*} = 1$ where g^* is the cometric), φ_t is the geodesic flow on S^*M and $\pi : S^*M \to M$ is the canonical projection.

Lemma 2. *There exists a continuous path of points $s \mapsto x_s \in M$, passing through x_0 at $s = 0$, such that, setting $\gamma_s(t) = \pi \circ \varphi_t(x_s, \xi_0)$, we have*

$$\lim_{s\to 0} \left(\chi_{\overline{\omega}}(\gamma_s(t)) + \chi_{\overline{\omega}}(\gamma_{-s}(t))\right) = 1 \tag{5}$$

for almost every $t \in [0,T]$ such that $\gamma(t) \in \partial\omega$.

Proof. To prove this fact, we assume that, in a local chart, $\gamma(t) = (t, 0, \ldots, 0)$. This is true at least in a neighborhood of $x_0 = \gamma(0) = 0$, and this holds true along $\gamma(\cdot)$ as long as there is no conjugate point. We also assume that, in this chart, any other geodesic ray starting at

$(0, x_2^0, \ldots, x_n^0)$ in a neighborhood of $\gamma(0) = (0, \ldots, 0)$ is given by $(t, x_2^0, \ldots, x_n^0)$ (projection onto M of the extremal field). This classical construction of the so-called *extremal field* can actually be done on any subinterval of $[0, T]$ along which there is no conjugate point. Note that the set of conjugate times along $[0, T]$ is of Lebesgue measure zero[1]. Let us search an appropriate $(n-1)$-tuple $(x_2^0, \ldots, x_n^0) \in \mathbb{R}^{n-1} \setminus \{0\}$ such that the family of points $x_s = (0, sx_2^0, \ldots, sx_n^0)$, $s \in (-1, 1)$, gives (5). Note that the geodesic ray starting at x_s is $\gamma_s(t) = (t, sx_2^0, \ldots, sx_n^0)$ in the local chart.

In what follows, we set $N = \partial\omega = \overline{\omega} \setminus \mathring{\omega} = \overline{\omega} \setminus \omega$ (ω is open). By assumption, ω is an embedded C^1 submanifold of M with boundary and one has $\dim N = \dim M - 1$. By assumption, in a neighborhood U of any point of N, the set $N \cap U$ is a codimension-one hypersurface of M, written as $F = 0$ with $F : U \to \mathbb{R}$ of class C^1, which is separating ω and $M \setminus \omega$ in the sense that $\omega \cap U = \{F < 0\}$, $N = \{F = 0\}$ and $M \setminus \omega = \{F \geqslant 0\}$.

It suffices to prove that, for almost every time t at which $\gamma_0(t) = \gamma(t) \in N$ and $\dot\gamma(t) \in T_{\gamma(t)}N$, the points $\gamma_s(t)$ and $\gamma_{-s}(t)$ are on different sides with respect to the (locally) separating manifold N for s small enough.

This is obvious when γ is transverse to N. We set $\Omega = \{t \in [0, T] \mid \gamma(t) \in N,\ \dot\gamma(t) \in T_{\gamma(t)}N\}$. It is a closed subset of $[0, T]$. Let $t \in \Omega$. In the local chart the tangent space $T_{\gamma(t)}N$ is an hyperplane of $\mathbb{R}^n$ containing the line $\mathbb{R}(1, 0, \ldots, 0)$. Its projection onto $\{0\} \times \mathbb{R}^{n-1}$ (the hyperplane orthogonal to the line $\gamma(\cdot)$) is an hyperplane of $\{0\} \times \mathbb{R}^{n-1}$, of normal vector $(0, v(t))$ with $v(t) \in \mathbb{R}^{n-1}$ of Euclidean norm 1. Since only the direction of $v(t)$ is important, we assume that $v(t) \in \mathbb{P}^{n-2}(\mathbb{R})$, the projective space.

We claim that:

There exists $V \in \mathbb{P}^{n-2}(\mathbb{R})$ such that $\langle V, v(t)\rangle \neq 0$ for almost every $t \in \Omega$.

With this result, setting $V = (x_2^0, \ldots, x_n^0)$, the points x_s defined above give the lemma.

Let us now prove the claim. We define $A = \{(t, V) \in \Omega \times \mathbb{P}^{n-2}(\mathbb{R}) \mid \langle V, v(t)\rangle = 0\}$. By definition, given $(t, V) \in \Omega \times \mathbb{P}^{n-2}(\mathbb{R})$ we have $\chi_A(t, V) = 1$ when $V \in v(t)^\perp$. Since $v(t)^\perp \cap \mathbb{P}^{n-2}(\mathbb{R})$ is of codimension one in $\mathbb{P}^{n-2}(\mathbb{R})$, we have $\int_{\mathbb{P}^{n-2}(\mathbb{R})} \chi_A(t, V)\, d\mathcal{H}^{n-2} = 0$ for every $t \in \Omega$, where we have endowed $\mathbb{P}^{n-2}(\mathbb{R})$ with the Hausdorff measure $\mathcal{H}^{n-2}$. Therefore, by the Fubini theorem,

$$0 = \int_\Omega \int_{\mathbb{P}^{n-2}(\mathbb{R})} \chi_A(t, V)\, d\mathcal{H}^{n-2}\, dt = \int_{\mathbb{P}^{n-2}(\mathbb{R})} \int_\Omega \chi_A(t, V)\, dt\, d\mathcal{H}^{n-2}$$

and thus $\int_\Omega \chi_A(t, V)\, dt = 0$ for almost every $V \in \mathbb{P}^{n-2}(\mathbb{R})$. Fixing such a V, it follows that $\chi_A(t, V) = 0$ for almost every $t \in \Omega$, and the claim is proved. $\square$

In view of proving Remark 1, note that the argument above still works in dimension 2 with ω piecewise C^1 (but not in dimension greater than or equal to 3: see the counterexample given in Section 1). In more general, in any dimension, the argument above still works if ω is such that, for almost every time t, the subdifferential at $\gamma(t)$ of $\partial\omega \cap \gamma(\cdot)^\perp$ is a singleton.

At this step, we have embedded the ray γ given by Lemma 1 into a family of rays γ_s which enjoy a kind of transversality property with respect to $N = \partial\omega$. Let us consider the partition

$$[0, T] = A_1 \cup A_2 \cup A_3$$

into three disjoint measurable sets, with

$$\begin{aligned} A_1 &= \{t \in [0, T] \mid \gamma(t) \in \omega\}, \\ A_2 &= \{t \in [0, T] \mid \gamma(t) \in M \setminus \overline{\omega}\}, \\ A_3 &= \{t \in [0, T] \mid \gamma(t) \in \partial\omega\}. \end{aligned}$$

[1] This is a general fact in Riemannian geometry. Indeed, a conjugate time is a time at which a non-zero Jacobi field vanishes. Since Jacobi fields are solutions of a second-order ordinary differential equation, such times must be isolated, for otherwise the Jacobi field would vanish at the second order and thus would be identically zero.

Since $\gamma_s(\cdot)$ converges uniformly to $\gamma(\cdot)$ as $s \to 0$ and since ω and $M \setminus \overline{\omega}$ are open, we have:

- $\lim_{s\to 0} \left(\chi_{\overline{\omega}}(\gamma_s(t)) + \chi_{\overline{\omega}}(\gamma_{-s}(t))\right) = 2$ for every $t \in A_1$;
- $\lim_{s\to 0} \left(\chi_{\overline{\omega}}(\gamma_s(t)) + \chi_{\overline{\omega}}(\gamma_{-s}(t))\right) = 0$ for every $t \in A_2$;
- $\lim_{s\to 0} \left(\chi_{\overline{\omega}}(\gamma_s(t)) + \chi_{\overline{\omega}}(\gamma_{-s}(t))\right) = 1$ for almost every $t \in A_3$ (by Lemma 2).

By the Lebesgue dominated convergence theorem, we infer that

$$\lim_{s\to 0} \int_0^T \left(\chi_{\overline{\omega}}(\gamma_s(t)) + \chi_{\overline{\omega}}(\gamma_{-s}(t))\right) dt = 2|A_1| + |A_3|.$$

Now, on the one part, by the first step we have $\frac{1}{T}|A_1| = \frac{1}{T}\int_0^T \chi_\omega(\gamma(t)) = \ell^T(\omega)$. On the other part, since A_1 and A_3 are disjoint we have $\frac{1}{T}(|A_1| + |A_3|) \leqslant 1$. Hence

$$\lim_{s\to 0} \frac{1}{T} \int_0^T \left(\chi_{\overline{\omega}}(\gamma_s(t)) + \chi_{\overline{\omega}}(\gamma_{-s}(t))\right) dt \leqslant \ell^T(\omega) + 1.$$

Since $\ell^T(\overline{\omega}) \leqslant \frac{1}{T}\int_0^T \chi_{\overline{\omega}}(\gamma_{\pm s}(t))dt$ for every s by definition, we infer that $2\ell^T(\overline{\omega}) \leqslant \ell^T(\omega) + 1$. Theorem 1 is proved.

3 Proof of Theorem 2

Theorem 2 states that, given any $T > 0$ and any $\varepsilon \in [0,1]$, the random variable $\ell^T(\omega_\varepsilon^n)$ converges in probability to ε as $n \to +\infty$, i.e.,

$$\lim_{n\to+\infty} \mathbb{P}\left(|\ell^T(\omega_\varepsilon^n) - \varepsilon| \geqslant \delta\right) = 0 \qquad \forall \delta > 0.$$

This section is organized as follows. We make a preliminary remark in Section 3.1. In Section 3.2, we give the successive steps of the proof, involving intermediate lemmas that are proved. One of the main ingredients of the proof of Theorem 2 is a large deviation property which is established in Section 3.3. In Section 3.4, we also provide a proof of Corollary 3.

3.1 Preliminaries

For any geodesic ray $\gamma \in \Gamma$ and any measurable subset $\omega \subset M$, we set

$$m_\gamma^T(\omega) = \frac{1}{T}\int_0^T \chi_\omega(\gamma(t))\, dt$$

so that we have

$$\ell^T(\omega) = \inf_{\gamma\in\Gamma} m_\gamma^T(\omega).$$

Lemma 3. *Given any measurable subset $\omega \subset M$, the mapping $\Gamma \ni \gamma \mapsto m_\gamma^T(\omega)$ is upper semi-continuous.*

Given any ω that is a union of squares from the grid $\mathcal{G}^n$ with n fixed, the mapping $\Gamma \ni \gamma \mapsto m_\gamma^T(\omega)$ is continuous at each γ which

- *is not horizontal or vertical;*
- *is horizontal and vertical but meets no corner.*

Proof. Upper semi-continuity in general is obvious. Let us prove the second part of the lemma. Let $(\gamma_k)_{k\in\mathbb{N}}$ be a sequence of geodesic rays converging to $\gamma \in \Gamma$. We assume that either that γ is not horizontal or is not vertical, or that γ is horizontal or vertical and meets no corner. Writing

$$\chi_\omega = \sum_{i,j=1}^{n} \tau(i,j)\chi_{c_{ij}}$$

where $\tau(i,j) = 1$ if $c_{ij} \subset \omega$ and $\tau(i,j) = 0$ otherwise, and noting that

$$m_{\gamma_k}^T(\omega) = \sum_{i,j=1}^{n} \tau(i,j) m_{\gamma_k}^T(c_{ij})$$

we have to prove that $m_{\gamma_k}^T(c_{ij})$ converges to $m_\gamma^T(c_{ij})$ as $k \to +\infty$. This follows from the dominated convergence theorem and from the fact that $(\chi_{c_{ij}}(\gamma_k))_{k\in\mathbb{N}}$ converges almost everywhere to $\chi_{c_{ij}}(\gamma)$. The latter claim can be shown by distinguishing between two cases: if $\gamma(t) \in \mathring{c}_{ij}$ then for k large enough we have $\gamma_k(t) \in \mathring{c}_{ij}$ and $\chi_{c_{ij}}(\gamma_k(t)) = \chi_{c_{ij}}(\gamma(t)) = 1$ for such k. The same conclusion occurs if $\gamma(t) \notin c_{ij}$. Since the set of points t such that $\gamma(t) \in \partial c_{ij}$ is finite (the speed of the geodesic ray is equal to 1), the lemma follows. □

Since $\gamma \mapsto m_\gamma^T(\omega)$ is only upper semi-continuous but not continuous, it will actually be convenient to slightly modify its definition when γ is horizontal (or vertical) and meets a corner by setting

$$m_\gamma^T(\omega) = \inf_{(\gamma'_k)_{k\in\mathbb{N}}} \liminf_{k\to+\infty} m_{\gamma'_k}^T(\omega) \tag{6}$$

where the infimum is taken on the set of sequences $(\gamma'_k)_{k\in\mathbb{N}}$ of horizontal (or vertical) geodesic rays converging to γ. With this modification, the mapping $\gamma \mapsto m_\gamma^T(\omega)$ is now lower semi-continuous and thus is continuous, and we still do have $\ell^T(\omega) = \inf_{\gamma\in\Gamma} m_\gamma^T(\omega)$ because the value of $m_\gamma^T(\omega)$ at possible points of discontinuity is changed into a value of the closure of the range of the mapping.

In particular, since the set of geodesic rays of length T is compact, the infimum in the definition of $\ell^T(\omega)$ is reached for some $\gamma \in \Gamma$, i.e., we have $\ell^T(\omega) = m_\gamma^T(\omega)$.

3.2 Proof of Theorem 2

For every $(i,j) \in \{1,\ldots,n\}^2$, let X_{ij} be the random variable equal to 1 whenever $c_{ij} \subset \omega_\varepsilon^n$ and 0 otherwise. Recall that, assuming that all square cells of the grid are initially white, when ranging over the grid, for each cell we randomly darken the cell with a probability ε (Bernoulli law) and in this case we set $X_{ij} = 1$; otherwise we let $X_{ij} = 0$. By construction, the random laws $(X_{ij})_{1\leqslant i,j\leqslant n}$ are independent and identically distributed (i.i.d.), of expectation ε and of variance $\varepsilon(1-\varepsilon)$.

Given any fixed geodesic ray $\gamma \in \Gamma$, we denote by $t_{ij}(\gamma)$ the time spent by γ in the square cell c_{ij}. We have $\sum_{i,j=1}^n t_{ij}(\gamma) = T$. It may happen that γ crosses several times (at most $T+1$ times) the square cell c_{ij}, and in this case $t_{ij}(\gamma)$ is the sum of several passage times; since the maximal time spent in c_{ij} by a ray in one passage is $\frac{\sqrt{2}}{n}$, we have $t_{ij}(\gamma) \leqslant (T+1)\frac{\sqrt{2}}{n}$. Summing up, we have

$$\forall (i,j) \in \{1,\ldots,n\}^2 \qquad \frac{t_{ij}(\gamma)}{T} \leqslant \frac{T+1}{T}\frac{\sqrt{2}}{n} \qquad \text{and} \qquad \sum_{i,j=1}^{n} \frac{t_{ij}(\gamma)}{T} = 1 \tag{7}$$

and we also note that

$$m_\gamma^T(\omega_\varepsilon^n) = \frac{1}{T}\int_0^T \chi_{\omega_\varepsilon^n}(\gamma(t))\,dt = \sum_{i,j=1}^{n} \frac{t_{ij}(\gamma)}{T} X_{ij}. \tag{8}$$

In particular, the random variable $m_\gamma^T(\omega_\varepsilon^n)$ is a weighted sum of independent Bernoulli laws, and thus

$$\mathbb{E}\, m_\gamma^T(\omega_\varepsilon^n) = \varepsilon \tag{9}$$

and

$$\operatorname{Var} m_\gamma^T(\omega_\varepsilon^n) = \sum_{i,j=1}^n \left(\frac{t_{ij}(\gamma)}{T}\right)^2 \varepsilon(1-\varepsilon) \leqslant \frac{T+1}{T}\frac{\sqrt{2}\varepsilon(1-\varepsilon)}{n} \tag{10}$$

where we have used (7) and the fact that $0 \leqslant \frac{t_{ij}(\gamma)}{T} \leqslant 1$.

To prove Theorem 2, we have to prove that, given any $T > 0$ and any $\varepsilon \in [0,1]$, for every $\delta > 0$ we have

(i) $\lim_{n\to+\infty} \mathbb{P}\left(\ell^T(\omega_\varepsilon^n) \leqslant \varepsilon + \delta\right) = 1$,

(ii) $\lim_{n\to+\infty} \mathbb{P}\left(\ell^T(\omega_\varepsilon^n) \geqslant \varepsilon - \delta\right) = 1$, or equivalently, $\lim_{n\to+\infty} \mathbb{P}\left(\ell^T(\omega_\varepsilon^n) \leqslant \varepsilon - \delta\right) = 0$.

Proof of (i). Using (8) and by definition of ℓ^T which is an infimum over all rays, we have

$$\ell^T(\omega_\varepsilon^n) \leqslant m_\gamma^T(\omega_\varepsilon^n) = \sum_{i,j=1}^n \frac{t_{ij}(\gamma)}{T} X_{ij}. \tag{11}$$

Applying the Bienaymé-Tchebychev inequality to the random variable $m_\gamma^T(\omega_\varepsilon^n)$, we have

$$\mathbb{P}\left(|m_\gamma^T(\omega_\varepsilon^n) - \mathbb{E}\, m_\gamma^T(\omega_\varepsilon^n)| \geqslant \delta\right) \leqslant \frac{\operatorname{Var} m_\gamma^T(\omega_\varepsilon^n)}{\delta^2}$$

and thus, using (9) and (10),

$$\lim_{n\to+\infty} \mathbb{P}\left(m_\gamma^T(\omega_\varepsilon^n) \leqslant \varepsilon + \delta\right) = 1. \tag{12}$$

Finally, (i) follows from (11) and (12).

Proof of (ii). Establishing (ii) is much more difficult. We proceed in several steps, by proving the following successive lemmas that are in order.

Lemma 4. *If (ii) is true for every $T \in (0,1)$, then it is true for every $T > 0$.*

Thanks to Lemma 4 (proved further), we now assume that $0 < T < 1$.

We introduce some notations. Let Γ^0, Γ^1 and Γ^2 be the sets of geodesic rays of $M = \mathbb{T}^2$ meeting respectively zero, at least one and at least two corners of the grid $\mathcal{G}^n$ (by definition, a corner is a point $(i/n, j/n)$ in $[0,1]^2$, for some $(i,j) \in \{0,\ldots,n\}^2$). Given any $\omega \subset M = \mathbb{T}^2$ that is the union of disjoint closed square cells of $\mathcal{G}^n$, we have

$$\ell^T(\omega) = \inf_{\gamma\in\Gamma^0\cup\Gamma^1} m_\gamma^T(\omega).$$

We also define

$$\ell^T_{\Gamma^1}(\omega) = \inf_{\gamma\in\Gamma^1} m_\gamma^T(\omega), \qquad \ell^T_{\Gamma^2}(\omega) = \inf_{\gamma\in\Gamma^2} m_\gamma^T(\omega).$$

Of course, we have

$$\ell^T(\omega) \leqslant \ell^T_{\Gamma^1}(\omega) \leqslant \ell^T_{\Gamma^2}(\omega).$$

Lemma 5. *We have* $|\ell^T(\omega_\varepsilon^n) - \ell^T_{\Gamma^1}(\omega_\varepsilon^n)| = \mathrm{O}(1/n)$ *as* $n \to +\infty$.

Lemma 6. *We have* $|\ell^T_{\Gamma^1}(\omega_\varepsilon^n) - \ell^T_{\Gamma^2}(\omega_\varepsilon^n)| = \mathrm{O}(1/n)$ *as* $n \to +\infty$.

Lemma 7. *We have* $\lim_{n\to+\infty} \mathbb{P}\left(\ell^T_{\Gamma^2}(\omega_\varepsilon^n) \leqslant \varepsilon - \delta\right) = 0$.

Finally, (ii) follows from the above lemmas, that are proved hereafter.

3.2.1 Proof of Lemma 4

Let $T \geqslant 1$ and let $m \in \mathbb{N}^*$ be such that $T' = \frac{T}{m} \in (0,1)$. Let us prove that

$$\ell^T(\omega_\varepsilon^n) \geqslant \ell^{T'}(\omega_\varepsilon^n). \tag{13}$$

Given any $\rho > 0$, let γ be a geodesic ray such that

$$\ell^T(\omega_\varepsilon^n) + \rho \geqslant \frac{1}{T}\int_0^T \chi_{\omega_\varepsilon^n}(\gamma(t))\,dt.$$

Setting $\gamma_k(\cdot) = \gamma(kT' + \cdot)$ for every $k \in \{0,\dots,m-1\}$, we have

$$\ell^T(\omega_\varepsilon^n) + \rho \geqslant \frac{1}{T}\int_0^T \chi_{\omega_\varepsilon^n}(\gamma(t))\,dt = \frac{1}{T}\sum_{k=0}^{m-1}\int_0^{T'} \chi_{\omega_\varepsilon^n}(\gamma_k(t))\,dt \geqslant \frac{1}{T}\sum_{k=0}^{m-1} T'\ell^{T'}(\omega_\varepsilon^n) = \ell^{T'}(\omega_\varepsilon^n)$$

Letting ρ tend to 0, we obtain (13).

Since $0 < T' < 1$, (ii) is true for this final time T', i.e.,

$$\lim_{n\to+\infty} \mathbb{P}\left(\ell^{T'}(\omega_\varepsilon^n) \geqslant \varepsilon - \delta\right) = 1.$$

Therefore, using (13), we obtain (ii) for the final time T.

3.2.2 Proof of Lemma 5

Let ω be a subset of $[0,1]^2$ that is a union of square cells of the grid $\mathcal{G}_n$. Recall that $\ell^T(\omega) \leqslant \ell^T_{\Gamma^1}(\omega)$. Now, let us prove that $\ell^T_{\Gamma^1}(\omega) \leqslant \ell^T(\omega) + \frac{C}{n}$ where $C > 0$ does neither depend on n nor on ω (which is a stronger statement, implying the lemma). To this aim, let $\gamma \in \Gamma$ be such that

$$m^T_\gamma(\omega) < \ell^T(\omega) + \frac{1}{nT}. \tag{14}$$

If $\gamma \in \Gamma^1$ then we are done. Hence, in what follows we assume that $\gamma \in \Gamma^0$. Moreover, since γ meets no corner, rotating slightly γ if necessary, by continuity of m^T_γ (see the modified definition (6) of m^T_γ in Section 3.1) we can assume that γ is neither vertical nor horizontal and still satisfies (14). Let $\mathbf{n}$ be a unit vector orthogonal to $\gamma'(0)$. For every $s \in \mathbb{R}$, we denote by $\mathcal{T}_{s\mathbf{n}}$ the translation of vector $s\mathbf{n}$ and we define the translated geodesic ray $\gamma_s = \mathcal{T}_{s\mathbf{n}} \circ \gamma$. By continuity, γ_s meets the same square cells as γ if $|s|$ is small enough. Let $I(\gamma)$ denote the subset of all indices $(i,j) \in \{1,\dots,n\}^2$ such that γ crosses the cell squares c_{ij}. If $|s|$ is small enough then

$$m^T_{\gamma_s}(\omega) = \sum_{(i,j)\in I(\gamma)} \frac{t_{ij}(\gamma_s)}{T} X_{ij}$$

where $t_{ij}(\gamma_s)$ is the time spent by γ_s in c_{ij}. Now, note that for given indices i and j such that $\gamma(0), \gamma(T) \notin \mathring{c}_{ij}$, an easy geometric argument shows that $s \mapsto t_{ij}(\gamma_s)$ is affine and nonconstant with respect to s. Hence, denoting by $I'(\gamma)$ the set of $(i,j) \in I(\gamma)$ such that $\gamma(0), \gamma(T) \notin \mathring{c}_{ij}$ (note that $\#I(\gamma) - 2 \leqslant \#I'(\gamma) \leqslant \#I(\gamma)$), we define

$$M_{\gamma_s}(\omega) = \frac{1}{T} \sum_{(i,j)\in I'(\gamma)} t_{ij}(\gamma_s) X_{ij}.$$

We have

$$M_{\gamma_s}(\omega) \leqslant m^T_{\gamma_s}(\omega) \leqslant M_{\gamma_s}(\omega) + \sum_{(i,j)\in I(\gamma)\setminus I'(\gamma)} \frac{t_{ij}(\gamma_s)}{T} X_{ij} \leqslant M_{\gamma_s}(\omega) + \frac{2\sqrt{2}}{nT}.$$

Since $t_{ij}(\gamma_s)$ is an affine function for $(i,j) \in I'(\gamma)$, so is $s \mapsto M_{\gamma_s}(\omega)$. Replacing s by $-s$ if necessary, we infer the existence of $s_0 > 0$ such that the mapping $s \mapsto M_{\gamma_s}(\omega)$ is decreasing on $(0, s_0)$ and moreover $\gamma_{s_0} \in \Gamma^1$. Since γ is neither vertical nor horizontal, so is γ_{s_0}. We then infer that

$$m^T_{\gamma_{s_0}}(\omega) \leqslant M_{\gamma_{s_0}}(\omega) + \frac{2\sqrt{2}}{nT} = \lim_{s\to s_0} M_{\gamma_s}(\omega) + \frac{2\sqrt{2}}{nT} \leqslant m^T_{\gamma}(\omega) + \frac{2\sqrt{2}}{nT}.$$

Using (14), we obtain

$$\ell^T_{\Gamma^1}(\omega) \leqslant m^T_{\gamma_{s_0}}(\omega) < \ell^T(\omega) + \frac{2\sqrt{2}+1}{nT}.$$

The conclusion follows.

3.2.3 Proof of Lemma 6

Let ω be a subset of $[0,1]^2$ that is a union of square cells of the grid $\mathcal{G}_n$. Recall that $\ell^T_{\Gamma^1}(\omega) \leqslant \ell^T_{\Gamma^2}(\omega)$. Now, let us prove that $\ell^T_{\Gamma^2}(\omega) \leqslant \ell^T_{\Gamma^1}(\omega) + \frac{C}{n}$ where $C > 0$ does neither depend on n nor on ω (this is a stronger statement, implying the lemma). To this aim, let $\gamma \in \Gamma^1 \setminus \Gamma^2$. It suffices to find $\gamma' \in \Gamma^2$ such that

$$m^T_{\gamma'}(\omega) \leqslant m^T_{\gamma}(\omega) + \frac{C}{n}. \tag{15}$$

By definition, the geodesic ray γ meets exactly one corner of the grid, that we denote by O. We denote by R_θ the rotation centered at O with angle θ and we define the geodesic ray $\gamma_\theta = r_\theta \circ \gamma$. As in the proof of Lemma 5, let $I'(\gamma)$ be the set of all indices $(i,j) \in \{1,\dots,n\}^2$ such that γ_θ crosses c_{ij} and $\gamma(0), \gamma(T) \notin \mathring{c}_{ij}$. This condition does not depend on $\theta \in J = (-\theta_-, \theta_+)$ where $\theta_-, \theta_+ > 0$ are such that $\gamma_\theta \notin \Gamma^2$ for every $\theta \in J$. As before, the reason why we require that $\gamma(0), \gamma(T) \notin \mathring{c}_{ij}$ is technical: it allows us to make easier the computation of the time $t_{ij}(\gamma_\theta)$ spent by γ_θ in c_{ij}. As in the proof of Lemma 5, defining

$$M_{\gamma_\theta}(\omega) = \frac{1}{T} \sum_{(i,j)\in I'(\gamma)} t_{ij}(\gamma_\theta) X_{ij} \tag{16}$$

where we recall that X_{ij} is equal to 1 if $c_{ij} \subset \omega$ and 0 otherwise, we have

$$M_{\gamma_\theta}(\omega) \leqslant m^T_{\gamma_\theta}(\omega) \leqslant M_{\gamma_\theta}(\omega) + \frac{2\sqrt{2}}{nT}. \tag{17}$$

We can assume that the real numbers $\theta_\pm$ above are such that J is "maximal", meaning that $\gamma_{\theta_\pm} \in \Gamma^2$. Now, we define the function f on J by

$$f(\theta) = M_{\gamma_\theta}(\omega)$$

and we extend it by continuity on $\bar{J}$. It remains to prove that that

$$\min(f(-\theta_-), f(\theta_+)) \leqslant \min_J f + \frac{C}{n} \tag{18}$$

where C does neither depend on n nor on γ. Indeed, assuming that (18) is satisfied, let us finish the proof of Lemma 6. Without loss of generality, we assume that $f(-\theta_-) \leqslant f(\theta_+)$. Note that the modified definition (6) of m_γ^T (in Section 3.1), (16) and (17) imply that $m^T_{\gamma_{\theta_-}}(\omega) \leqslant f(-\theta_-) + \frac{2\sqrt{2}}{nT}$. This inequality is actually an equality unless γ_{θ_-} is horizontal or vertical. Therefore

$$m^T_{\gamma_{\theta_-}}(\omega) \leqslant \min_J f + \frac{C}{n} + \frac{2\sqrt{2}}{nT}.$$

Since $\min_J f \leqslant f(0) = M_\gamma(\omega) \leqslant m^T_\gamma(\omega)$, we obtain $m^T_{\gamma_{\theta_-}}(\omega) \leqslant m^T_\gamma(\omega) + \frac{C'}{n}$ with $C' = C + \frac{2\sqrt{2}}{T}$. Since $\gamma_{\theta_-} \in \Gamma^2$, the inequality (15) is satisfied and therefore Lemma 6 is proved.

It remains to prove (18). We first compute f by providing an explicit expression of $t_{ij}(\gamma_\theta)$. We claim that, if $(i,j) \in I$, then there exists $(a_{ij}, b_{ij}) \in \{-n,\dots,n\}^2$ such that

$$t_{ij}(\gamma_\theta) = \frac{a_{ij}}{n\sin(\theta_0+\theta)} + \frac{b_{ij}}{n\cos(\theta_0+\theta)} \tag{19}$$

where θ_0 is the angle between the horizontal axis and the geodesic ray γ. Without loss of generality we assume that $\theta_0 \in [0, \pi/2]$. Note that $\theta_0 \in (0, \pi/2)$, because otherwise the two conditions $\gamma \in \Gamma_1$ and $\theta_0 \in \{0, \pi/2\}$ (i.e., γ is horizontal or vertical) would imply that $\gamma \in \Gamma_2$, which is false. This also implies that $J \subset (0, \pi/2)$. Using the notations of Figure 4 where the corner O is assimilated

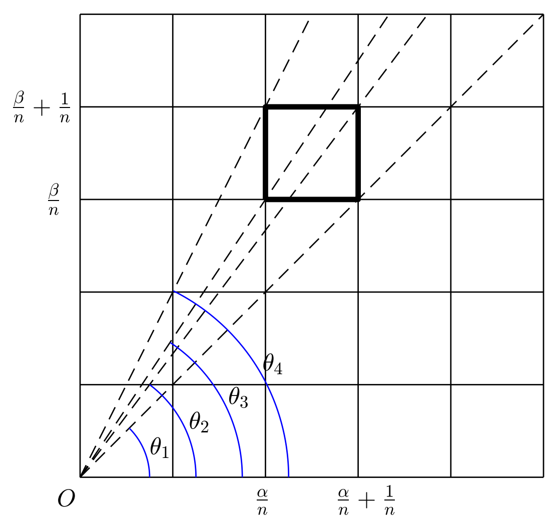

Figure 4: Particular geodesics issued from O and meeting the square with bold boundary.

to the origin, and focusing on the square c_{ij} whose coordinates are $(\alpha/n, \beta/n)$, $(\alpha/n + 1/n, \beta/n)$,

$(\alpha/n + 1/n, \beta/n + 1/n)$ and $(\alpha/n, \beta/n + 1/n)$, an elementary geometric reasoning gives

$$t_{ij}(\gamma_\theta) = \begin{cases} \frac{\alpha+1}{n\cos\theta} - \frac{\beta}{n\sin\theta} & \text{if } \theta_0 + \theta \in [\theta_1, \theta_2] \\ \frac{1}{n\sin\theta} & \text{if } \theta_0 + \theta \in [\theta_2, \theta_3] \\ \frac{\beta+1}{n\sin\theta} - \frac{\alpha}{n\cos\theta} & \text{if } \theta_0 + \theta \in [\theta_3, \theta_4] \end{cases}$$

where $0 < \theta_1 < \theta_2 < \theta_3 < \theta_4 < \pi/2$ are the angles between the horizontal axis and the particular geodesics passing at the corners of c_{ij} (see Figure 4). Note that this formula is still satisfied when $\theta_1 = 0$ or $\theta_4 = \pi/2$. In such cases, either α or β vanishes. The claim (19) is then proved.

Note that the rotation R_θ can be applied until the geodesic ray γ_θ meets a new corner. It is then easy to see that

$$\theta_+, \theta_- \leqslant \frac{C}{n} \tag{20}$$

where C does not depend on n and γ.

According to (19), we have

$$M_{\gamma_\theta}(\omega) = \frac{X_1}{\sin(\theta_0 + \theta)} + \frac{X_2}{\cos(\theta_0 + \theta)}, \tag{21}$$

with

$$X_1 = \frac{1}{n} \sum_{(i,j) \in I'(\gamma)} a_{ij} M_{\gamma_\theta}(c_{ij}) \quad \text{and} \quad X_2 = \frac{1}{n} \sum_{(i,j) \in I'(\gamma)} b_{ij} M_{\gamma_\theta}(c_{ij}),$$

for $\theta \in J$. Moreover,

$$|X_1| \leqslant \# I'(\gamma) \leqslant 2nT \quad \text{and} \quad |X_2| \leqslant \# I'(\gamma) \leqslant 2nT$$

because $\max(|a_{ij}|, |b_{ij}|) \leqslant nT$ and $M_{\gamma_\theta}(c_{ij}) \leqslant \frac{\sqrt{2}}{nT}$.

We are now in a position to establish (18). It is is obvious if $\min_J f$ is reached at θ_-or θ_+. Let us then assume that f reaches its minimum; without loss of generality, we assume that the minimum is reached at $\theta = 0$. In particular,

$$\left.\frac{d}{d\theta}\right|_{\theta=0} M_{\gamma_\theta}(\omega) = 0$$

which, according to (21), gives $\frac{\cos(\theta_0)}{\sin^2(\theta_0)} X_1 = \frac{\sin(\theta_0)}{\cos^2(\theta_0)} X_2$. From (21), we also have $f(0) = \frac{X_1}{\sin\theta_0} + \frac{X_2}{\cos\theta_0}$ and thus $X_1 = f(0)\sin^3(\theta_0)$ and $X_2 = f(0)\cos^3(\theta_0)$, so that

$$f(\theta) = M_{\gamma_\theta}(\omega) = f(0) \left(\frac{\sin^3(\theta_0)}{\sin(\theta_0 + \theta)} + \frac{\cos^3(\theta_0)}{\cos(\theta_0 + \theta)} \right).$$

Now,

$$f(\theta) - f(0) = f(0) \left(\sin^3\theta_0 \left(\frac{1}{\sin(\theta_0 + \theta)} - \frac{1}{\sin\theta_0} \right) + \cos^3\theta_0 \left(\frac{1}{\cos(\theta_0 + \theta)} - \frac{1}{\cos\theta_0} \right) \right).$$

This computation shows that $\theta_0 + \theta_+ \neq \pi/2$ or $\theta_0 - \theta_- \neq 0$. Otherwise, since $\theta_0 \notin \{0, \pi/2\}$, f would be not bounded on J when θ tends to $\theta_0 + \theta_+$ or $\theta_0 - \theta_-$, which is wrong by definition of f. This implies that

$$\arcsin\left(\frac{1}{nT}\right) \leqslant \theta_0 - \theta_- < \theta_0 + \theta_+ \leqslant \frac{\pi}{2} - \arcsin\left(\frac{1}{nT}\right). \tag{22}$$

Indeed, set $\gamma_- = \gamma_{\theta_0-\theta_-}$. We know that $\gamma_-([0,T])$ contains at least two corners and is not horizontal. This means that its second coordinate (in $\mathbb{R}^2$) increases at least of $1/n$ between 0 and T and hence $T\sin(\theta_0-\theta_-) \geqslant 1/n$. The same argument allows to prove the last inequality of (22).

From now on, we let n tend to $+\infty$. In particular, θ_0 depends on n. Let us prove that

$$f(\theta_-) - \min_J f = f(\theta_-) - f(0) \leqslant \frac{C}{n} \tag{23}$$

A similar argument will show that $f(\theta_+) - \min_J f \leqslant \frac{C}{n}$. These two estimates imply (18) and hence complete the proof of Lemma 6. It thus remains to prove (23). By the mean value theorem, there exists $\tilde{\theta} \in (0, \theta_-)$ such that

$$\frac{1}{\sin(\theta_0-\theta_-)} - \frac{1}{\sin\theta_0} = \frac{-\cos(\theta_0-\tilde{\theta})}{\sin^2(\theta_0-\tilde{\theta})}\theta_-.$$

Since $0 \leqslant \tilde{\theta} \leqslant \theta_-$, using that $|\sin(\theta_0)\cos(\theta_0-\tilde{\theta})| \leqslant 1$, we get from (20) that

$$\sin^3(\theta_0)\left|\frac{1}{\sin(\theta_0-\theta_-)} - \frac{1}{\sin\theta_0}\right| \leqslant \frac{C}{n}\left(\frac{\sin(\theta_0)}{\sin(\theta_0-\tilde{\theta})}\right)^2.$$

Now, by the mean value theorem,

$$\frac{\sin(\theta_0)}{\sin(\theta_0-\tilde{\theta})} = 1 + \frac{\sin(\theta_0)-\sin(\theta_0-\tilde{\theta})}{\sin(\theta_0-\tilde{\theta})} \leqslant 1 + \frac{\mathrm{O}(1/n)}{\sin(\theta_0-\tilde{\theta})}.$$

Using (22), there exists $C' > 0$ which does neither depend on γ nor on n such that $\frac{\mathrm{O}(1/n)}{\sin(\theta_0-\tilde{\theta})} = \mathrm{O}(1)$. This implies that

$$\sin^3(\theta_0)\left(\frac{1}{\sin(\theta_0-\tilde{\theta})} - \frac{1}{\sin\theta_0}\right) = \mathrm{O}\left(\frac{1}{n}\right).$$

Following the same reasoning, we prove that

$$\cos^3(\theta_0)\left(\frac{1}{\cos(\theta_0-\tilde{\theta})} - \frac{1}{\cos\theta_0}\right) = \mathrm{O}\left(\frac{1}{n}\right).$$

Actually, this last estimate is much easier since $|\cos(\theta_0)| \leqslant C\cos(\theta_0-\theta_-)$. This leads to (23) and concludes the proof of Lemma 6.

3.2.4 Proof of Lemma 7

Let $(i_1, j_1), (i_2, j_2) \in \{1, \ldots, n\}^2$ be such that the two square cells $c_{i_1j_1}$ and $c_{i_2j_2}$ of $\mathcal{G}^n$ are disjoint and let x_1 and x_2 be two corners of the grid $\mathcal{G}^n$. Any geodesic ray of length T such that

$$\gamma(0) \in c_{i_1j_1}, \quad \gamma(T) \in c_{i_2j_2} \quad \text{and} \quad x_i \in \gamma([0,T]),\ i = 1, 2,$$

will be denoted by $\gamma_{c_{i_1j_1}, c_{i_2j_2}, x_1, x_2}$.

Let $\hat{\Gamma}$ be the set of all such geodesic rays $\gamma_{c_{i_1j_1}, c_{i_2j_2}, x_1, x_2}$, as (i_1, j_1), (i_2, j_2), x_1 and x_2 vary. By construction, we have $\#\hat{\Gamma}^4 = \mathrm{O}(n^8)$.

Let $\hat{\gamma} \in \hat{\Gamma}$. By the large deviation result established in Proposition 1 in Section 3.3 (it can be applied thanks to (7)), we have

$$\mathbb{P}(m_{\hat{\gamma}}(\omega_\varepsilon^n) \leqslant \varepsilon - \delta) \leqslant C_{\varepsilon,\delta} e^{-\frac{m\delta^2}{c}}$$

where m is the number of square cells met by $\hat{\gamma}$. Let us evaluate m.

Since the diagonal length of each square cell c_{ij} is $\sqrt{2}/n$, the length of γ is bounded above by the length of the longest geodesic passing through m squares diagonally arranged, i.e., $\sqrt{2}m/n$. Therefore $m \geqslant nT/\sqrt{2}$ and

$$\mathbb{P}(m_{\hat{\gamma}}(\omega_\varepsilon^n) \leqslant \varepsilon - \delta) \leqslant C_{\varepsilon,\delta} e^{-\frac{nT\delta^2}{\sqrt{2}c}}.$$

Now, let $\gamma \in \Gamma^2$. By definition, there exist $(i_1, j_1), (i_2, j_2) \in \{1, \ldots, n\}^2$ and two corners of the grid $\mathcal{G}^n$ denoted by x_1 and x_2 such that $\gamma(0) \in c_{i_1 j_1}$, $\gamma(T) \in c_{i_2 j_2}$ and $x_i \in \gamma([0, T])$ for $i = 1, 2$. It follows that γ and $\gamma_{c_{i_1 j_1}, c_{i_2 j_2}, x_1, x_2}$ coincide everywhere, except maybe on the first and last squares. Therefore, since the diagonal length of a square cell of $\mathcal{G}^n$ is $\sqrt{2}/n$, we have

$$\left| m_\gamma^T(\omega_\varepsilon^n) - m_{\gamma_{c_{i_1 j_1}, c_{i_2 j_2}, x_1, x_2}}^T(\omega_\varepsilon^n) \right| \leqslant \frac{2\sqrt{2}}{n}.$$

We infer that $\inf_{\gamma \in \hat{\Gamma}} m_\gamma^T(\omega_\varepsilon^n) = \ell_{\Gamma^2}^T(\omega_\varepsilon^n) + \mathrm{O}(1/n)$. Therefore, if n is large enough then

$$(\ell_{\Gamma^2}^T(\omega_\varepsilon^n) \leqslant \varepsilon - \delta) \subset \left(\inf_{\gamma \in \hat{\Gamma}} m_\gamma^T(\omega_\varepsilon^n) \leqslant \varepsilon - 2\delta/3 \right)$$

and thus there exists $C > 0$ (not depending on n) such that

$$\begin{aligned}
\mathbb{P}\left(\ell_{\Gamma^2}^T(\omega_\varepsilon^n) \leqslant \varepsilon - \delta\right) \leqslant \mathbb{P}\left(\inf_{\gamma \in \hat{\Gamma}} m_\gamma^T(\omega_\varepsilon^n) \leqslant \varepsilon - 2\delta/3 \right) &= \mathbb{P}\left(\bigcup_{\gamma \in \hat{\Gamma}} \left(m_\gamma^T(\omega_\varepsilon^n) \leqslant \varepsilon - 2\delta/3 \right) \right) \\
&\leqslant \sum_{\gamma \in \hat{\Gamma}} \mathbb{P}\left(m_\gamma^T(\omega_\varepsilon^n) \leqslant \varepsilon - 2\delta/3 \right) \leqslant C n^8 e^{-\frac{4nT\delta^2}{9\sqrt{2}c}}
\end{aligned}$$

since $\#\hat{\Gamma} = \mathrm{O}(n^8)$. The conclusion follows.

3.3 A large deviation result for triangular arrays of Bernoulli variables

Proposition 1. *Let $m \geqslant 3$ be an integer and let $(\lambda_1, \ldots, \lambda_m)$ be a m-tuple of nonnegative real numbers satisfying $\sum_{i=1}^m \lambda_i = 1$ and $\lambda_i \leqslant \frac{c}{m}$ for every i, for some $c > 1$. We set $Y_m = \sum_{i=1}^m \lambda_i X_i$, where $(X_1, \ldots, X_m)$ is a m-tuple of i.i.d. random variables with expectation $\varepsilon \in [0, 1]$. Then, for every $\delta > 0$, there exists $C_{\varepsilon,\delta} > 0$ such that*

$$\mathbb{P}(|Y_m - \varepsilon| \geqslant \delta) \leqslant C_{\varepsilon,\delta} e^{-\delta^2 m/c}. \tag{24}$$

Remark 2. The assumptions that $c > 1$ and $m \geqslant 3$ are not restrictive. They allow us to provide a sharp estimate of the right-hand side of (24), by solving an auxiliary optimization problem.

Proof of Proposition 1. We have

$$\mathbb{P}(|Y_m - \varepsilon| \geqslant \delta) = \mathbb{P}\left(Y_m \geqslant \varepsilon + \delta\right) + \mathbb{P}\left(-Y_m \geqslant -\varepsilon + \delta\right)$$

and it suffices to prove that

$$\mathbb{P}(Y_m \geqslant \varepsilon + \delta) \leqslant C_{\varepsilon,\delta} e^{-\delta^2 m/c}$$

because the estimate on $\mathbb{P}\left(-Y_m \geqslant -\varepsilon + \delta\right)$ can be obtained similarly.

Let $s > 0$ to be chosen later. By the Markov inequality, we have

$$\mathbb{P}(Y_m \geqslant \varepsilon + \delta) = \mathbb{P}\left(e^{sm(Y_m - \varepsilon - \delta)} \geqslant 1 \right) \leqslant \mathbb{E}\left(e^{sm(Y_m - \varepsilon - \delta)} \right) = e^{-sm(\delta + \varepsilon)} \mathbb{E}\left(e^{smY_m} \right).$$

Using the definition of Y_m and independence of the random variables X_i, we infer that

$$\begin{aligned}\mathbb{P}(Y_m \geqslant \varepsilon+\delta) \leqslant e^{-sm(\delta+\varepsilon)} \prod_{i=1}^{m} \mathbb{E}\left(e^{sm\lambda_i X_i}\right) &= e^{-sm(\delta+\varepsilon)} \prod_{i=1}^{m} \left(1-\varepsilon+\varepsilon e^{sm\lambda_i}\right) \\ &= e^{-sm(\delta+\varepsilon)} F(\lambda_1,\ldots,\lambda_m) \leqslant e^{-sm(\delta+\varepsilon)} \max_{\Sigma_m} F \quad (25)\end{aligned}$$

where

$$F(\lambda_1,\ldots,\lambda_m) = \prod_{i=1}^{m} \left(1-\varepsilon+\varepsilon e^{sm\lambda_i}\right)$$

and

$$\Sigma_m = \left\{ (\lambda_1,\cdots,\lambda_m) \in \left[0,\frac{c}{m}\right]^m \mid \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

Lemma 8. *We have*

$$\max_{\Sigma_m} F = (1-\varepsilon+\varepsilon e^{sc})^{\lfloor \frac{m}{c} \rfloor} \left(1-\varepsilon+\varepsilon e^{s\left(m-c\lfloor \frac{m}{c}\rfloor\right)}\right)$$

where $\lfloor \cdot \rfloor$ *denotes the floor function.*

Proof of Lemma 8. Let $(a_1,\ldots,a_m) \in \Sigma_m$ be a point at which the continuous function F reaches its maximum over the compact set Σ_m. Let $(j,k) \in \{1,\ldots,m\}^2$ be such that $j \neq k$. We define the function α on $\mathbb{R}$ by

$$\alpha(u) = F(a_1,\ldots,a_j+u,\ldots,a_k-u,\ldots,a_m).$$

Setting $I_{j,k} = [\max(-a_j, a_k - c/m), \min(a_k, c/m - a_j)]$, for every $u \in I_{j,k}$ (i.e., $0 \leqslant a_j + u \leqslant c/m$ and $0 \leqslant a_k - u \leqslant c/m$, we have $(a_1,\ldots,a_j+u,\ldots,a_k-u,\ldots,a_m) \in \Sigma_m$. Note that $0 \in I_{j,k}$ and that, since $(a_1,\ldots,a_m)$ is a maximizer of F, we have $\alpha(u) \leqslant \alpha(0)$ for every $u \in I_{j,k}$. We have two possible cases:

(i) $a_j = 0$ or $a_j = c/m$ or $a_k = 0$ or $a_k = c/m$;

(ii) $0 < a_j < c/m$ and $0 < a_m < c/m$.

In the latter case (ii), we must have $\alpha'(0) = 0$, and since, by computing this derivative, we have

$$\alpha'(0) = sm\varepsilon(1-\varepsilon)\left(e^{sma_j} - e^{sma_k}\right) \prod_{\substack{1\leqslant i\leqslant m \\ i\neq j,\ i\neq k}} \left(1-\varepsilon+\varepsilon e^{sma_i}\right),$$

it follows that $a_j = a_k$.

Since the pair (j,k) of distinct integers was arbitrary, we conclude that there exists $\bar\lambda \in (0, c/m)$ such that $a_j \in \{0, \bar\lambda, c/m\}$ for every $j \in \{1,\ldots,m\}$. Let J be the set of indices such that $a_j = \bar\lambda$ for every $j \in J$. Denote by F_J the restriction of F to the set of all $(\lambda_1,\ldots,\lambda_m) \in \Sigma_m$ such that $\lambda_i = a_i$ for every $i \notin J$. Observing that F_J is the product of separate variables positive strictly convex functions, its Hessian $d^2F_J(a_1,\ldots,a_m)$ must be positive definite. But, by maximality of $(a_1,\ldots,a_m)$ and since $\sum_{i=1}^m a_i = 1$, $d^2F_J(a_1,\ldots,a_m)$ has at most one positive eigenvalue. Therefore J contains at most one element.

Noting that F is invariant under permutations, we infer that, performing a permutation of variables if necessary, there exists $N \in \{1,\ldots,m-2\}$ such that

$$a_i = \frac{c}{m} \quad \forall i \in \{1,\ldots,N\} \qquad \text{and} \qquad a_j = 0 \quad \forall j \in \{N+1,\ldots,m-1\}$$

and $a_m = 1 - Nc/m$ because $\sum_{i=1}^m a_i = 1$. Since $a_m \geqslant 0$, we must have $N \leqslant \min(\lfloor \frac{m}{c} \rfloor, m-2) = \lfloor \frac{m}{c} \rfloor$ because $c > 1$ and $m \geqslant 3$. Therefore

$$\max_{\Sigma_m} F = e^{\varphi(N)} \quad \text{with} \quad \varphi(x) = x \ln(1 - \varepsilon + \varepsilon e^{sc}) + \ln(1 - \varepsilon + \varepsilon e^{s(m-cx)}).$$

It remains to determine the best possible integer $N \in \{1, \dots, \lfloor \frac{m}{c} \rfloor\}$. We have

$$\varphi'(N) = \ln(1 - \varepsilon + \varepsilon e^{sc}) - \frac{sc\varepsilon e^{s(m-cN)}}{1 - \varepsilon + \varepsilon e^{s(m-cN)}} \geqslant \frac{sc\varepsilon(1-\varepsilon)}{1 - \varepsilon + \varepsilon e^{s(m-cN)}}(1 - e^{s(m-cN)}) > 0,$$

where we have used that $1 - \varepsilon + \varepsilon e^{s(m-cN)} > 1$ and $\ln(1 - \varepsilon + \varepsilon e^{sc}) \geqslant \varepsilon sc$ by concavity of the logarithm. Hence, the best integer N is $N = \lfloor \frac{m}{c} \rfloor$. The result follows. □

Using (25), Lemma 8 and the inequalities $m - c \lfloor \frac{m}{c} \rfloor \leqslant c$ and $1 < 1 - \varepsilon + \varepsilon e^{sc} \leqslant e^{sc}$, we obtain

$$\begin{aligned}\mathbb{P}(Y_m \geqslant \varepsilon + \delta) &\leqslant e^{-sm(\delta+\varepsilon)}(1 - \varepsilon + \varepsilon e^{sc})^{\lfloor \frac{m}{c} \rfloor + 1} \leqslant e^{-sm(\delta+\varepsilon)}(1 - \varepsilon + \varepsilon e^{sc})^{\frac{m}{c}+1} \\ &\leqslant (1 - \varepsilon + \varepsilon e^{sc}) e^{-m(cs(\varepsilon+\delta) - \ln(1-\varepsilon+\varepsilon e^{cs}))/c} \leqslant e^{sc} e^{-m\varphi_{\varepsilon,\delta}(cs)/c} \end{aligned} \tag{26}$$

where

$$\varphi_{\varepsilon,\delta}(u) = u(\varepsilon + \delta) - \ln(1 - \varepsilon + \varepsilon e^u).$$

In order to choose adequately the parameter s, we will use the following result.

Lemma 9. *For every $\delta > 0$, we have*

$$\inf_{u>0} \varphi_{\varepsilon,\delta}(u) < \delta^2 < \sup_{u>0} \varphi_{\varepsilon,\delta}(u).$$

Moreover, there exists $\hat{u}_{\varepsilon,\delta} \geqslant 0$ such that $\varphi_{\varepsilon,\delta}(\hat{u}_{\varepsilon,\delta}) = \delta^2$, and $\hat{u}_{\varepsilon,\delta} \leqslant U_{\varepsilon,\delta}$ where

$$U_{\varepsilon,\delta} = \begin{cases} \frac{\delta^2}{\varepsilon+\delta-1} & \text{if } \delta > 1 - \varepsilon, \\ \ln\left(\frac{(1-\varepsilon)e^{\delta^2}}{1-\varepsilon e^{\delta^2}}\right) & \text{if } \delta = 1 - \varepsilon, \\ \ln\left(\frac{(1-\varepsilon)(\varepsilon+\delta)}{\varepsilon(1-\varepsilon-\delta)}\right) & \text{if } \delta < 1 - \varepsilon. \end{cases}$$

Proof of Lemma 9. We distinguish between several cases:

- If $\delta > 1-\varepsilon$, using that $\ln(1-\varepsilon+\varepsilon e^u) \leqslant u$ for every $u \geqslant 0$, it follows that $\varphi_{\varepsilon,\delta}(u) \geqslant (\varepsilon+\delta-1)u$ and hence
$$\sup_{u>0} \varphi_{\varepsilon,\delta}(u) \geqslant \varphi_{\varepsilon,\delta}\left(\frac{\delta^2}{\varepsilon+\delta-1}\right) \geqslant \delta^2.$$
Moreover, since $\varphi_{\varepsilon,\delta}(0) = 0$, by continuity of $\varphi_{\varepsilon,\delta}$, there exists $\hat{u}_{\varepsilon,\delta} \in (0, \delta^2/\varepsilon + \delta - 1)$ such that $\varphi_{\varepsilon,\delta}(\hat{u}_{\varepsilon,\delta}) = \delta^2$.

- If $\delta = 1 - \varepsilon$, since $\varphi_{\varepsilon,\delta}$ is monotone increasing and $\varphi_{\varepsilon,\delta}(u) \leqslant \lim_{u\to+\infty} \varphi_{\varepsilon,\delta}(u) = -\ln \varepsilon$, we obtain
$$\sup_{u>0} \varphi_{\varepsilon,\delta}(u) = -\ln(1-\delta) \geqslant \delta \geqslant \delta^2.$$
Moreover, noting that $\varphi_{\varepsilon,\delta}(u) = \ln\left(\frac{e^u}{1-\varepsilon+\varepsilon e^u}\right)$, $\hat{u}_{\varepsilon,\delta}$ can be computed and we obtain the formula stated in Lemma 9.

- If $\delta < 1-\varepsilon$, we set $u_{\varepsilon,\delta} = \ln\left(\frac{(\varepsilon+\delta)(1-\varepsilon)}{\varepsilon(1-\varepsilon-\delta)}\right)$. A straightforward study shows that $\varphi_{\varepsilon,\delta}$ is monotone increasing on $[0, u_{\varepsilon,\delta}]$ and monotone decreasing on $[u_{\varepsilon,\delta}, +\infty)$ and then

$$\sup_{u>0} \varphi_{\varepsilon,\delta}(u) = (\varepsilon+\delta)\ln\left(\frac{(\varepsilon+\delta)(1-\varepsilon)}{\varepsilon(1-\varepsilon-\delta)}\right) - \ln\left(\frac{1-\varepsilon}{1-\varepsilon-\delta}\right)$$
$$= (\varepsilon+\delta)\ln\left(\frac{\varepsilon+\delta}{\varepsilon}\right) + (\varepsilon+\delta-1)\ln\left(\frac{1-\varepsilon}{1-\varepsilon-\delta}\right).$$

Setting $a = \varepsilon+\delta \in (\varepsilon, 1)$, we get that

$$\sup_{u>0} \varphi_{\varepsilon,\delta}(u) = f(a) \quad \text{with} \quad f(a) = a\ln\left(\frac{a}{\varepsilon}\right) + (1-a)\ln\left(\frac{1-a}{1-\varepsilon}\right).$$

We compute

$$f'(a) = \ln\left(\frac{a}{\varepsilon}\right) - \ln\left(\frac{1-a}{1-\varepsilon}\right) \quad \text{and} \quad f''(a) = \frac{1}{a(1-a)}.$$

Note that $f''(a) \geqslant 4$ for every $a \in (0,1)$. Integrating two times this inequality and using that $f(\varepsilon) = f'(\varepsilon) = 0$, we get $f(a) \geqslant 2\delta^2$ and therefore

$$\sup_{u>0} \varphi_{\varepsilon,\delta}(u) \geqslant 2\delta^2 > \delta^2.$$

□

We choose s such that $cs = \hat{u}_{\varepsilon,\delta}$, where $\hat{u}_{\varepsilon,\delta}$ is given by Lemma 9, and we set $C(\varepsilon,\delta) = e^{\hat{u}_{\varepsilon,\delta}}$. The inequality (26) yields

$$\mathbb{P}(Y_m \geqslant \varepsilon+\delta) \leqslant C_{\varepsilon,\delta} e^{-m\delta^2/c}.$$

Proposition 1 is proved. □

3.4 Proof of Corollary 3

It is proved in [6] that $\ell^T(\omega) \leqslant |\omega|$ for every Riemann integrable subset ω of $M = \mathbb{T}^2$. Let $\delta > 0$ be arbitrary. Let us apply Theorem 2 by changing the parameter ε of the Bernoulli law to $\varepsilon' = \varepsilon - \delta$. Noting that the random variable $|\omega^n_{\varepsilon'}|$ follows the law $\frac{1}{n^2}B(n^2, \varepsilon')$, we obtain

$$\lim_{n\to+\infty} \mathbb{P}(|\omega^n_{\varepsilon'}| \leqslant \varepsilon'+\delta) = \lim_{n\to+\infty} \mathbb{P}(|\omega^n_{\varepsilon'}| \leqslant \varepsilon) = 1$$

and

$$\lim_{n\to+\infty} \mathbb{P}(\ell^T(\omega^n_{\varepsilon'}) \geqslant \varepsilon'-\delta) = \lim_{n\to+\infty} \mathbb{P}(\ell^T(\omega^n_{\varepsilon'}) \geqslant \varepsilon-2\delta) = 1.$$

As a consequence,

$$\lim_{n\to+\infty} \mathbb{P}(\ell^T(\omega^n_{\varepsilon'}) \leqslant \varepsilon-2\delta \text{ or } |\omega^n_{\varepsilon'}| \geqslant \varepsilon) \leqslant \lim_{n\to+\infty} \mathbb{P}(\ell^T(\omega^n_{\varepsilon'}) \leqslant \varepsilon-2\delta) + \lim_{n\to+\infty} \mathbb{P}(|\omega^n_{\varepsilon'}| \geqslant \varepsilon) = 0$$

and therefore,

$$\lim_{n\to+\infty} \mathbb{P}(\ell^T(\omega^n_{\varepsilon'}) \geqslant \varepsilon-2\delta \text{ and } |\omega^n_{\varepsilon'}| \leqslant \varepsilon) = 1,$$

which yields the existence of a domain ω that is a union of subsquares of a grid $\mathcal{G}^n$ for some $n \in \mathbb{N}^*$ such that $\ell^T(\omega) \geqslant \varepsilon - 2\delta$ and $|\omega| \leqslant \varepsilon$. We then infer that $\varepsilon - 2\delta \leqslant \sup_{|\omega|\leqslant\varepsilon} \ell^T(\omega) = \varepsilon$ and we conclude since δ has been chosen arbitrarily. Corollary 3 is proved.

A Some properties of the functional ℓ^T

Recall that, given any $T > 0$ and any Lebesgue measurable subset ω of M, denoting by χ_ω the characteristic function of ω, we have defined

$$\ell^T(\omega) = \inf_{\gamma\in\Gamma} \frac{1}{T}\int_0^T \chi_\omega(\gamma(t))\,dt.$$

The functional ℓ^T can be extended by replacing χ_ω by any measurable function a on M. It can even be extended further: any geodesic ray $\gamma \in \Gamma$ is the projection onto M of a geodesic curve on S^*M, that is, $\gamma(t) = \pi \circ \varphi_t(z)$ for some $z \in S^*M$. Here, w denote by $(\varphi_t)_{t\in\mathbb{R}}$ the Riemannian geodesic flow, where, for every $t \in \mathbb{R}$, φ_t is a symplectomorphism on (T^*M, ω) which preserves S^*M. Now, given any bounded measurable function a on (S^*M, μ_L) and given any $T > 0$, we define

$$\ell^T(a) = \inf_{z\in S^*M} \frac{1}{T}\int_0^T a\circ\varphi_t(z)\,dt = \inf_{z\in S^*M} \bar{a}_T(z)$$

where $\bar{a}_T(z) = \frac{1}{T}\int_0^T a\circ\varphi_t(z)\,dt$ and where the unit cotangent bundle S^*M is endowed with the Liouville measure μ_L. Note that $\ell^T(a) = \ell^T(a\circ\varphi_t)$, i.e., ℓ^T is invariant under the geodesic flow.

It can also be noted that for a fixed the function $T \mapsto T\ell^T(a)$ is superadditive.

Of course, we recover the initial definition of ℓ^T by pushforward to M under the canonical projection $\pi : S^*M \to M$: given any bounded measurable function f on (M, dx_g), we have

$$(\pi_*\ell^T)(f) = \ell^T(\pi^*f) = \ell^T(f\circ\pi) = \inf_{z\in S^*M}\frac{1}{T}\int_0^T f\circ\pi\circ\varphi_t(z)\,dt = \inf_{\gamma\in\Gamma}\frac{1}{T}\int_0^T f(\gamma(t))\,dt$$

that we simply denote by $\ell^T(f)$. When $f = \chi_\omega$, we recover $\ell^T(\omega)$.

Remark 3. Setting $a_t = a\circ\varphi_t$, and assuming that $a \in C^0(S^*M)$ is the principal symbol of a pseudo-differential operator $A \in \Psi^0(M)$ (of order 0), that is, $a = \sigma_P(A)$, we have, by the Egorov theorem (see [10]),

$$a_t = a\circ\varphi_t = \sigma_P(A_t) \qquad \text{with} \qquad A_t = e^{-it\sqrt{\triangle}}Ae^{it\sqrt{\triangle}}$$

where $\sigma_P(\cdot)$ is the principal symbol. Accordingly, we have $\bar{a}_T = \sigma_P(\bar{A}_T)$ with

$$\bar{A}_T = \frac{1}{T}\int_0^T A_t\,dt = \frac{1}{T}\int_0^T e^{-it\sqrt{\triangle}}Ae^{it\sqrt{\triangle}}\,dt.$$

We provide hereafter a microlocal interpretation of the functionals ℓ^T and we give a relationship with the wave observability constant.

Microlocal interpretation of ℓ^T and of the wave observability constant. Let f_T be such that $\hat{f}_T(t) = \frac{1}{T}\chi_{[0,T]}(t)$, i.e., $f_T(t) = \frac{1}{2\pi}ie^{iTt/2}\mathrm{sinc}(Tt/2))$. Note that $\int_{\mathbb{R}}\hat{f}_T = 1$, i.e., equivalently, $f_T(0) = 1$. We denote by X the Hamiltonian vector field on S^*M of the geodesic flow (we have $e^{tX} = \varphi_t$ for every $t \in \mathbb{R}$), and we define the selfadjoint operator $S = \frac{X}{i}$. Using that $a\circ e^{tX} = (e^{tX})^*a = e^{tL_X}a = e^{itS}a$, we get

$$\ell^T(a) = \inf_{z\in S^*M}\frac{1}{T}\int_0^T a\circ e^{tX}(z)\,dt = \inf_{z\in S^*M}\int_{\mathbb{R}}\hat{f}_T(t)e^{itS}a\,dt\ (z) = \inf f_T(S)a.$$

Besides, setting $A = \mathrm{Op}(a)$ (where Op is a quantization), we have

$$\bar{A}_T(a) = \frac{1}{T}\int_0^T e^{-it\sqrt{\triangle}} a e^{it\sqrt{\triangle}}\, dt = \int_{\mathbb{R}} \hat{f}_T(t) e^{-it\sqrt{\triangle}} a e^{it\sqrt{\triangle}}\, dt = A_{f_T} = \sum_{\lambda,\mu} f_T(\lambda-\mu) P_\lambda A P_\mu .$$

Restricting to half-waves, the wave observability constant is therefore given (see [7]) by

$$C_T(a) = \inf_{\|y\|=1} \langle \bar{A}_T(a) y, y\rangle = \inf_{\|y\|=1} \langle A_{f_T} y, y\rangle.$$

Note that

$$\begin{aligned}\langle A_{f_T} y, y\rangle = \sum_{\lambda,\mu} f_T(\lambda-\mu) \langle A P_\lambda y, P_\mu y\rangle = \sum_{\lambda,\mu} f_T(\lambda-\mu) \int_M a\, P_\lambda y\, \overline{P_\mu y} \\ = \sum_{\lambda,\mu} f_T(\lambda-\mu) a_\lambda \bar{a}_\mu \int_M a \phi_\lambda \phi_\mu\end{aligned}$$

and we thus recover the expression of $C_T(a)$ by series expansion (see [7]).

Note also that, as said before, the principal symbol of $A_{f_T} = \bar{A}_T(a)$ is

$$\sigma_P(A_{f_T}) = \sigma_P(\bar{A}_T(a)) = a_{f_T} = \int_{\mathbb{R}} \hat{f}_T(t) a\circ e^{tX}\, dt = f_T(S)a$$

and thus

$$\ell^T(a) = \inf \sigma_P(\bar{A}_T(a)).$$

Semi-continuity properties of ℓ^T. Note the obvious fact that if a and b are functions such that $a \leqslant b$ and for which the following quantities make sense, then $\ell^T(a) \leqslant \ell^T(b)$. In other words, the functional ℓ^T is nondecreasing.

Lemma 10. *Let ω be a subset of M, let $T > 0$ be arbitrary and let $(h_k)_{k\in\mathbb{N}^*}$ be a uniformly bounded sequence of Borel measurable functions on M. If h_k converges pointwise to χ_ω, then*

$$\limsup_{k\to+\infty} \ell^T(h_k) \leqslant \ell^T(\omega)$$

and if moreover $\chi_\omega \leqslant h_k$ for every $k \in \mathbb{N}^*$, then

$$\ell^T(\omega) = \lim_{k\to+\infty} \ell^T(h_k) = \inf_{k\in\mathbb{N}^*} \ell^T(h_k).$$

Proof. Let $\gamma \in \Gamma$ be arbitrary. By pointwise convergence, we have $h_k(\gamma(t)) \to \chi_\omega(\gamma(t))$ for every $t \in [0,T]$, and it follows from the dominated convergence theorem that $\ell^T(h_k) \leqslant \frac{1}{T}\int_0^T h_k(\gamma(t))\, dt \to \frac{1}{T}\int_0^T \chi_\omega(\gamma(t))\, dt$, and thus $\limsup_{k\to+\infty} \ell^T(h_k) \leqslant \frac{1}{T}\int_0^T \chi_\omega(\gamma(t))\, dt$. Since this inequality is valid for any $\gamma \in \Gamma$, the first inequality follows.

If moreover $\chi_\omega \leqslant h_k$ then $\limsup_{k\to+\infty} \ell^T(h_k) \leqslant \ell^T(\omega) \leqslant \ell^T(h_k)$ and the result follows. $\square$

Note that, in the above proof, we use the fact that $h_k(x) \to \chi_\omega(x)$ for *every* x. Almost everywhere convergence (in the Lebesgue sense) would not be enough.

Remark 4. We denote by d the geodesic distance on (M,g). It is interesting to note that, given any subset ω of M:

- ω is open if and only if there exists a sequence of continuous functions h_k on M satisfying $0 \leqslant h_k \leqslant h_{k+1} \leqslant \chi_\omega$ for every $k \in \mathbb{N}^*$ and converging pointwise to χ_ω.

  Indeed, if ω is open, then one can take for instance $h_k(x) = \min(1, k\, d(x, \omega^c))$. Conversely, since $h_k(x) = 0$ for every $x \in \omega^c$, by continuity of h_k it follows that $h_k(x) = 0$ for every $x \in \overline{\omega^c} = (\mathring{\omega})^c$. Now take $x \in \omega \setminus \mathring{\omega}$. We have $h_k(x) = 0$, and $h_k(x) \to \chi_\omega(x)$, hence $\chi_\omega(x) = 0$ and therefore $x \in \omega^c$. Hence ω is open.

- ω is closed if and only if there exists a sequence of continuous functions h_k on M satisfying $0 \leqslant \chi_\omega \leqslant h_{k+1} \leqslant h_k \leqslant 1$ for every $k \in \mathbb{N}^*$ and converging pointwise to χ_ω.

  Indeed, if ω is closed, then one can take $h_k(x) = \max(0, 1 - k\, d(x, \omega))$. Conversely, since $h_k(x) = 1$ for every $x \in \omega$, by continuity of h_k it follows that $h_k(x) = 1$ for every $x \in \overline{\omega}$, and thus $\chi_{\overline{\omega}} \leqslant h_k \leqslant 1$. Now take $x \in \overline{\omega} \setminus \omega$. We have $h_k(x) = 1$ and $h_k(x) \to \chi_\omega(x)$, hence $\chi_\omega(x) = 1$ and therefore $x \in \omega$. Hence ω is closed.

Lemma 11. *Let ω be an open subset of M and let $T > 0$ be arbitrary. For every sequence of continuous functions h_k on M converging pointwise to χ_ω, satisfying moreover $0 \leqslant h_k \leqslant h_{k+1} \leqslant \chi_\omega$ for every $k \in \mathbb{N}^*$, we have*

$$\ell^T(\omega) = \lim_{k \to +\infty} \ell^T(h_k) = \sup_{k \in \mathbb{N}^*} \ell^T(h_k).$$

Proof. Since $h_k \leqslant \chi_\omega$, we have $\ell^T(h_k) \leqslant \ell^T(\omega)$. By continuity of h_k and by compactness of geodesics, there exists a geodesic ray γ_k such that $\ell^T(h_k) = \frac{1}{T}\int_0^T h_k(\gamma_k(t))\, dt$. Again by compactness of geodesics, up to some subsequence γ_k converges to a ray $\bar\gamma$ in $C^0([0,T], M)$. We claim that

$$\liminf_{k \to +\infty} h_k(\gamma_k(t)) \geqslant \chi_\omega(\bar\gamma(t)) \qquad \forall t \in [0,T].$$

Indeed, either $\bar\gamma(t) \notin \omega$ and then $\chi_\omega(\bar\gamma(t)) = 0$ and the inequality is obviously satisfied, or $\bar\gamma(t) \in \omega$ and then, using that ω is open, for k large enough we have $\gamma_k(t) \in U$ where $U \subset \omega$ is a compact neighborhood of $\bar\gamma(t)$. Since h_k is monotonically nondecreasing and χ_ω is continuous on U, it follows from the Dini theorem that h_k converges uniformly to χ_ω on U, and then we infer that $h_k(\gamma_k(t)) \to 1 = \chi_\omega(\bar\gamma(t))$. The claim is proved. Now, we infer from the Fatou lemma that

$$\begin{aligned} \ell^T(\omega) \leqslant \frac{1}{T}\int_0^T \chi_\omega(\bar\gamma(t))\, dt \leqslant \frac{1}{T}\int_0^T \liminf_{k \to +\infty} h_k(\gamma_k(t))\, dt \\ \leqslant \liminf_{k \to +\infty} \frac{1}{T}\int_0^T h_k(\gamma_k(t))\, dt = \liminf_{k \to +\infty} \ell^T(h_k) \leqslant \ell^T(\omega) \end{aligned}$$

and we get the equality. □

Remark 5. The results of Lemmas 10 and 11 are valid as well for subsets of S^*M (which is a metric space).

Lemma 12. *Let ω be an open subset of M and let $T > 0$ be arbitrary. There exists $\gamma \in \Gamma$ such that $\ell^T(\omega) = \frac{1}{T}\int_0^T \chi_\omega(\gamma(t))\, dt$, i.e., the infimum in the definition of $\ell^T(\omega)$ is reached.*

Proof. The argument is almost contained in the proof of Lemma 11, but for completeness we give the detail. Let $(\gamma_k)_{k \in \mathbb{N}^*}$ be a sequence of rays such that $\frac{1}{T}\int_0^T \chi_\omega(\gamma_k(t))\, dt \to \ell^T(\omega)$. By compactness of geodesics, $\gamma_k(\cdot)$ converges uniformly to some ray $\gamma(\cdot)$ on $[0,T]$.

Let $t \in [0,T]$ be arbitrary. If $\gamma(t) \in \omega$ then for k large enough we have $\gamma_k(t) \in \omega$, and thus $1 = \chi_\omega(\gamma(t)) \leqslant \chi_\omega(\gamma_k(t)) = 1$. If $\gamma(t) \in M \setminus \omega$ then $0 = \chi_\omega(\gamma(t)) \leqslant \chi_\omega(\gamma_k(t))$ for any k. In all cases, we have obtained the inequality $\chi_\omega(\gamma(t)) \leqslant \liminf_{k \to +\infty} \chi_\omega(\gamma_k(t))$, for every $t \in [0,T]$.

By the Fatou lemma, we infer that

$$\ell^T(\omega) \leqslant \frac{1}{T}\int_0^T \chi_\omega(\gamma(t))\,dt \leqslant \frac{1}{T}\int_0^T \liminf_{k\to+\infty} \chi_\omega(\gamma_k(t))\,dt \leqslant \liminf_{k\to+\infty} \frac{1}{T}\int_0^T \chi_\omega(\gamma_k(t))\,dt = \ell^T(\omega)$$

and the equality follows. □

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